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Preferences and Similarities

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INTERNATIONAL CENTRE FOR MECHANICAL SCIENCES

COURSES AND LECTURES - No. 504

PREFERENCES AND SIMILARITIES

EDITED BY

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PREFACE

This volume entitled "Preferences and Similarities" and read at the 8th International Workshop of the International School for the Synthesis of Expert Knowledge (ISSEK), Udine, Italy, October, 5th–7th, 2006, contains thirteen papers. All papers were thoroughly reviewed by the scientific program committee after they were presented, and were carefully prepared for publication. As its preceding ones, this workshop was hosted by the Centre International des Sciences Mécaniques (CISM), and was held in the picturesque Palazzo del Torso, Udine.

The workshop was jointly organised by Professors G. Della Riccia, University of Udine, D. Dubois, CNRS and University of Toulouse, R. Kruse, University of Magdeburg, and H.-J. Lenz, Freie Universität Berlin. As the workshop was an invitational one, the four organisers invited international research workers with a significant contribution to the field of interest. This volume focuses on preference and similarity, issues that have gained much attention in various scientific communities over the last couple of years. It is worthwhile mentioning, that the scope of similarity and preference is still broadening due to the exploration of new fields of application. This is caused by the strong impact of vagueness, imprecision, uncertainty and dominance on human and agent information, communication, planning, decision, action, and control as well as by the technical progress of the information technology itself. The subject is equally of interest to computer scientists, statisticians, operations researchers, experts in AI, cognitive psychologists and economists. Areas of applications may include robotics, database and information retrieval, agent and decision theory, data analysis or data mining. This fact is furthermore evident from the strongly increasing influence of communication and co-operation in electronic markets using the Internet. Information is expected to be available in time, at every site, personalized, and disseminated to privileged users irrespective of their hardware devices, *cf. SOA, and Web Services. Planning agents and guides are yet other representatives of this development. If data or information from several sources is integrated into a single database and embedded into a context, the figures can be interpreted and utilised as knowledge for planning, decision-making or control. In all these cases preferences on decision alternatives, and distances or similarities between pairs of objects measured by corresponding variables are of main interest.*

Chapter 1: Similarity, Dominance, Fuzzy Logic and Efficiency. De Baets, Bernard: Similarity of Fuzzy Sets and Dominance of Random Variables: A Quest for Transitivity. The author puts the focus on the occurrence of various types of transitivity in two relational frameworks for expressing similarities and preferences in a quantitative way. The first framework is that of fuzzy relations where transitivity is defined by means of a general conjunction operation. He discusses two approaches to the measurement of similarity of fuzzy sets: a logical approach based on bi-residual operators and a cardinal approach based on fuzzy set cardinalities. The second framework is that of reciprocal relations; the corresponding notion of transitivity is cycle-transitivity, a symmetric form of transitivity. It plays a crucial role in the description of different types of transitivity arising in the comparison of random variables in terms of winning probabilities. Ruspini, Enrique: A Logic-Based View of Similarities and Preferences. After recalling basic concepts of the interpretation of fuzzy logic in terms of metrics in a finite set of states, he considers the nature of the information required to generate the underlying metrics. He concludes that similarity measures are typically derived from preference relations. The relation between similarity measures and utility functions is used to extend his similarity-based approach to marginal and conditional preferences. Some examples motivate basic requirements for a comprehensive logic-based approach to a calculus of similarity and preferences. Such a formalism allows considering the relative desirability of attaining potential system states from the perspective of different preference criteria. Bonnefon, Jean-François, Dubois, Didier and Fargier, Hélène: An overview of bipolar qualitative decision rules. People often evaluate decisions by listing their positive and negative features. The problem is then to compare such sets. Assuming bipolarity of evaluations and qualitative ratings, they present and axiomatically characterise some decision rules based on *the idea of focusing on the most salient features, that are capable of handling positive and negative affects. The simplest are extensions of the maximin and maximax criteria to the bipolar case but they suffer from a lack of discrimination power. In order to overcome this weakness of the decision rules, refinements are proposed, capturing both the Pareto-efficiency principle and the order-of-magnitude reasoning principle of neglecting less important criteria. The most decisive rule uses a lexicographic ranking of the pros and cons. This turns out to be a special kind of the Cumulative Prospect Theory, and subsumes the "Take the best" heuristic.*

Chapter 2: Uncertainty, Vagueness, Incompleteness, Truthlikeliness and Proximity. Godo, Luis and Rodríguez, Ricardo O.: Logical approaches to similarity-based reasoning: An overview. The paper surveys different approaches to formalize similarity-based reasoning mainly from the view point of a graded notion of truthlikeliness. The difference between the traditional concepts of uncertainty and vagueness and truthlikeliness is explained. Fuzzy similarity relations are used to model truthlikeliness with a graded notion. They follow up an approach, that may be named semantically oriented, which considers a similarity relation on set of possible worlds rather than on propositions. This approach has its roots in work on approximate truth by Ruspini. Finally, the authors analyse non-monotonic similarity-based reasoning. The idea is to consider various kinds of similarity-based orderings in order to define non-monotonic consequence relations and operators for revision. Golińska-Pilarek and Orlowska, Ewa: Logics of similarity and their dual Tableaux. A Survey. The authors survey qualitative similarity models for information systems based on databases from the standpoint of rough sets. Their formal approach is relevant for information systems with incomplete data and uncertainty of knowledge. Their concept of similarity relations includes a qualitative degree of similarity as well as the relevant context. In an axiomatic way they present modal logics characterized by the classes of relational systems based on a subset of those similarity relations. A relational inference system for those logics is based on dual tableaux. Relational proof theory enables the authors to establish a proof system for non-classical logics—represented as a deduction rule set—in a modular way. Lenz, Hans-J.: Proximities in Statistics: Similarity and Distance. The author surveys similarity and distance measures *used in statistics for clustering, classification, or multi-dimensional scaling etc. Such pairwise relations are fulfilling conditions like symmetry and reflexivity. Special attention is paid to the type of scales of a variable (attribute), i.e. nominal (often binary), ordinal, metric (interval and ratio), and mixed types of scales. The paper considers the algebraic structure of proximities as suggested by Hartigan (1967) and Cormack (1971), information-theoretic measures as introduced by Jardine and Sibson (1971), and a probabilistic measure as proposed by Skarabis (1970) in more detail. This W-distance not only measures as usual the proximity between pairs of observations in a given finite dimensional data space but allows to establish a preorder on pairs of observations based upon the corresponding probability distribution. This makes it possible to discriminate even between two pairs of objects that have the same distance value but strongly differ with respect to the likelihood of the observations.*

Chapter 3: Similarity, Independence, Probability and Game Theory. Klawonn, Frank and Kruse, Rudolf: Similarity Relations and Independence Concepts. The paper focuses on similarity relations, their connection to fuzzy systems, and the inherent independence assumption that are implicitly taken in models using possibility theory, belief functions etc. Motivated by the sound definition of independence in statistics some approaches to distance-based similarity relations are proposed and analysed. The results give evidence of strong differences between the independence concept used in probability theory and useful for similarity relations. Sudkamp, Tom: Imprecision and Structure in Modelling Subjective Similarity. This paper generalises feature-based similarity to gain flexibility in modelling subjective similarity judgements. The presence-absence taxonomic feature approach is extended to attributes that take partial membership or fuzzy sets as values. The minimum specificity principle is applied to obtain possibilistic bounds on the combination of similarity values. Priority and bipolarity are added to model inter-object relationships and constraints in similarity judgements. Shafer, Glenn: Defensive Forecasting. The theory of defensive forecasting uses game theory for the notation of probability thus replacing measure theory by game theory. This approach allows to prove a classical theorem of probability theory such as the law of large numbers by a betting strategy that multiplies the capital at risk by a large factor if the theorem's prediction fails. Defensive forecast-

ing first identifies a strategy that succeeds if the probabilistic forecasts are inaccurate and then makes forecasts that will defeat this strategy. Both betting strategy and forecasts are based on the similarity of the current and previous situation.

Chapter 4: Argument-based Decision Making, Qualitative Preferences Reasoning, and Label Rankings. Amgoud, Leila and Prade, Henri: Comparing decisions on the basis of a bipolar typology of arguments. The authors consider argument-based decision-making. They pick up their former proposal of a typology with eight types of arguments instead of just one, i.e. pro or con. They emphasize the bipolar nature of selecting alternatives, i.e. by explicit considering prioritised goals and rejections that are certainly or possibly to be avoided. Decisions can be attacked or supported by arguments, and have a status. The logic properties of this argumentative framework are presented. Domshlak, Carmel: A Snapshot on Reasoning with Qualitative preference Statements in AI. The paper is devoted to the interpretation and formal reasoning about sets of qualitative preference statements—generally not complete—in the context of ordinal preferences of a decision maker. The author sketches a general scheme for reasoning about user preferences that unifies the treatment of this cognitive paradigm in an analogous way as done in the field of artificial intelligence. As up to now no requirements are specified for such systems in any specific context, the author votes for more interaction between academics and practioneers. Hüllermeier, Eyke and Fürnkranz, Johannes: Learning Preference Models from Data: On the Problem of Label Rankings and its Variants. In label ranking, the problem is to learn a ranking function that maps from an instance space to rankings over a finite set of labels. A ranking function thus defined can be considered as a generalization of a conventional classification function. To solve the label ranking problems the authors propose an approach based on the idea of ranking by pairwise comparison (RPC). Having a learning sample at hand, first of all, a binary preference relation is induced by applying a generalisation of pairwise classification. Then a ranking is derived, by transforming the preference relation. This procedure can be adapted to different loss functions simply by selecting different ranking procedures. Therefore it gives much flexibility. A related ranking procedure called "ranking through iterated choice" is experimentally investigated. Rossi, Francesca: Constraints and

Preferences: Modelling Frameworks and Multi-Agent Settings. The author aims for a unifying formalism to model preferences and constraints and manage them efficiently. The preferences she considers are of various kinds: qualitative/quantitative, marginal/conditional, negative/positive. The constraints may be soft or hard. She reviews existing formalisms for representing both entities, especially soft constraints and CP nets. Voting theory comes in when multi agent preference aggregation is considered. Several semantics for preference aggregation are proposed, and notions such as incompleteness and non-manipulability. Arrow's famous theorem on "Fairness" shows theoretical limitations for any formalism.

The editors of this volume are very thankful to all our authors for re-submitting their papers, and Mrs. Angelika Wnuk, Freie Universität Berlin, for her diligent work as a workshop convenor. We would like to thank the following institutions for substantial help on various levels:

- The International School for the Synthesis of Expert Knowledge (ISSEK) again for promoting the workshop.
- The University of Udine for administrative support.
- The Centre International des Sciences Mécaniques (CISM) for hosting a group of enthusiastic people with a common interest in preferences, similarities and cappuccino.

On behalf of all participants we express our deep gratitude to FONDAZIONE CASSA di RISPARMIO di UDINE e PORDENONE for their financial support of our participants.

FONDAZIONE CRUP

CASSA DI RISPARMIO DI UDINE E PORDENONE

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5th April, 2008

CONTENTS

Chapter 4: Argument-based Decision Making, Qualitative Preferences Reasoning, and Label Rankings

Similarity of Fuzzy Sets and Dominance of Random Variables: a Quest for Transitivity

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Abstract We present several relational frameworks for expressing similarities and preferences in a quantitative way. The main focus is on the occurrence of various types of transitivity in these frameworks. The first framework is that of fuzzy relations; the corresponding notion of transitivity is C-transitivity, with C a conjunctor. We discuss two approaches to the measurement of similarity of fuzzy sets: a logical approach based on biresidual operators and a cardinal approach based on fuzzy set cardinalities. The second framework is that of reciprocal relations; the corresponding notion of transitivity is cycle-transitivity. It plays a crucial role in the description of different types of transitivity arising in the comparison of random variables in terms of winning probabilities.

1 Introduction

Comparing objects in order to group together similar ones or distinguish better from worse is inherent to human activities in general and scientific disciplines in particular. In this overview paper, we present some relational frameworks that allow to express the results of such a comparison in a numerical way, typically by means of numbers in the unit interval. A first framework is that of fuzzy relations and we discuss how it can be used to develop cardinality-based, i.e. based on the counting of features, similarity measurement techniques. A second framework is that of reciprocal relations and we discuss how it can be used to develop methods for comparing random variables. Rationality considerations demand the presence of some kind of transitivity. We therefore review in detail the available notions of transitivity and point out where they occur.

This chapter is organised as follows. In Section 2, we present the two relational frameworks mentioned, the corresponding notions of transitivity and the connections between them. In Section 3, we explore the framework of fuzzy relations and its capacity for expressing the similarity of fuzzy sets. Section 4 is dedicated to the framework of reciprocal relations and its potential for the development of methods for the comparison of random variables. We wrap up in Section 5 with a short conclusion.

2 Relational frameworks and their transitivity

2.1 Fuzzy relations

Transitivity is an essential property of relations. A (binary) *relation* R on a universe X (the *universe of discourse* or the *set of alternatives*) is called *transitive* if for any $(a,b,c) \in X^3$ it holds that $(a,b) \in R \wedge (b,c) \in R$ implies $(a,c) \in R$. Identifying R with its characteristic mapping, *i.e.* defining $R(a,b) = 1$ if $(a,b) \in R$, and $R(a,b) = 0$ if $(a,b) \notin R$, transitivity can be stated equivalently as $R(a,b) = 1 \wedge R(b,c) = 1$ implies $R(a,c) = 1$. Other equivalent formulations may be devised, such as

$$(R(a,b) \geq \alpha \wedge R(b,c) \geq \alpha) \Rightarrow R(a,c) \geq \alpha\,, \tag{1}$$

for any $\alpha \in]0,1]$. Transitivity can also be expressed in the following functional form

$$\min(R(a,b), R(b,c)) \leq R(a,c)\,. \tag{2}$$

Note that on $\{0,1\}^2$ the minimum operation is nothing else but the Boolean conjunction.

A *fuzzy relation* R on X is an $X^2 \to [0,1]$ mapping that expresses the degree of relationship between elements of X: $R(a,b) = 0$ means a and b are not related at all, $R(a,b) = 1$ expresses full relationship, while $R(a,b) \in]0,1[$ indicates a partial degree of relationship only. In fuzzy set theory, formulation (2) has led to the popular notion of T-transitivity, where a t-norm is used to generalize Boolean conjunction. A binary operation $T : [0,1]^2 \to [0,1]$ is called a *t-norm* if it is increasing in each variable, has neutral element 1 and is commutative and associative. The three main continuous t-norms are the minimum operator $T_{\mathbf{M}}$, the algebraic product $T_{\mathbf{P}}$ and the Łukasiewicz t-norm $T_{\mathbf{L}}$ (defined by $T_{\mathbf{L}}(x,y) = \max(x+y-1,0)$). For an excellent monograph on t-norms and t-conorms, we refer to Klement et al. (2000).

However, we prefer to work with a more general class of operations called conjunctors. A *conjunctor* is a binary operation $C : [0,1]^2 \to [0,1]$ that is increasing in each variable and coincides on $\{0,1\}^2$ with the Boolean conjunction.

Definition 2.1. Let C be a conjunctor. A fuzzy relation R on X is called C-transitive if for any $(a,b,c) \in X^3$ it holds that

$$C(R(a,b), R(b,c)) \leq R(a,c)\,. \tag{3}$$

Interesting classes of conjunctors are the classes of semi-copulas, quasi-copulas, copulas and t-norms. *Semi-copulas* are nothing else but conjunctors with neutral element 1 (Durante and Sempi (2005)). Where t-norms have the additional properties of commutativity and associativity, quasi-copulas have the 1-Lipschitz property (Genest et al. (1999); Nelsen (1998)). A *quasi-copula* is a semi-copula that is *1-Lipschitz*: for any $(x, y, u, v) \in [0, 1]^4$ it holds that $|C(x, u) - C(y, v)| \leq |x - y| + |u - v|$. If instead of 1-Lipschitz continuity, C satisfies the *moderate growth property* (also called 2-monotonicity): for any $(x, y, u, v) \in [0, 1]^4$ such that $x \leq y$ and $u \leq v$ it holds that $C(x, v) + C(y, u) \leq C(x, u) + C(y, v)$, then C is called a *copula*.

Any copula is a quasi-copula, and therefore has the 1-Lipschitz property; the converse is not true. It is well known that a copula is a t-norm if and only if it is associative; conversely, a t-norm is a copula if and only if it is 1-Lipschitz. The t-norms $T_{\mathbf{M}}$, $T_{\mathbf{P}}$ and $T_{\mathbf{L}}$ are copulas as well. For any quasi-copula C it holds that $T_{\mathbf{L}} \leq C \leq T_{\mathbf{M}}$. For an excellent monograph on copulas, we refer to Nelsen (1998).

2.2 Reciprocal relations

Another interesting class of $X^2 \to [0, 1]$ mappings is the class of *reciprocal relations* Q (also called *ipsodual relations* or *probabilistic relations*) satisfying $Q(a, b) + Q(b, a) = 1$, for any $a, b \in X$. For such relations, it holds in particular that $Q(a, a) = 1/2$. Reciprocity is linked with completeness: let R be a complete ($\{0, 1\}$-valued) relation on X, which means that $\max(R(a, b), R(b, a)) = 1$ for any $a, b \in X$, then R has an equivalent $\{0, 1/2, 1\}$-valued reciprocal representation Q given by $Q(a, b) = 1/2(1 + R(a, b) - R(b, a))$.

Stochastic transitivity Transitivity properties for reciprocal relations rather have the logical flavor of expression (1). There exist various kinds of stochastic transitivity for reciprocal relations (David (1963); Monjardet (1988)). For instance, a reciprocal relation Q on X is called *weakly stochastic transitive* if for any $(a, b, c) \in X^3$ it holds that $Q(a, b) \geq 1/2 \wedge Q(b, c) \geq 1/2$ implies $Q(a, c) \geq 1/2$, which corresponds to the choice of $\alpha = 1/2$ in (1). In De Baets et al. (2006a), the following generalization of stochastic transitivity was proposed.

Definition 2.2. Let g be an increasing $[1/2, 1]^2 \to [0, 1]$ mapping such that $g(1/2, 1/2) \leq 1/2$. A reciprocal relation Q on X is called g-stochastic transitive if for any $(a, b, c) \in X^3$ it holds that

$$(Q(a,b) \geq 1/2 \ \wedge \ Q(b,c) \geq 1/2) \ \Rightarrow \ Q(a,c) \geq g(Q(a,b), Q(b,c)) \,.$$

Note that the condition $g(1/2, 1/2) \leq 1/2$ ensures that the reciprocal representation Q of any transitive complete relation R is always g-stochastic transitive. In other words, g-stochastic transitivity generalizes transitivity of complete relations. This definition includes the standard types of stochastic transitivity (Monjardet (1988)):

(i) *strong* stochastic transitivity when $g = \max$;

(ii) *moderate* stochastic transitivity when $g = \min$;

(iii) *weak* stochastic transitivity when $g = 1/2$.

In De Baets et al. (2006a), also a special type of stochastic transitivity has been introduced.

Definition 2.3. Let g be an increasing $[1/2, 1]^2 \to [0, 1]$ mapping such that $g(1/2, 1/2) = 1/2$ and $g(1/2, 1) = g(1, 1/2) = 1$. A reciprocal relation Q on X is called g-isostochastic transitive if for any $(a, b, c) \in X^3$ it holds that

$$(Q(a,b) \geq 1/2 \ \wedge \ Q(b,c) \geq 1/2) \ \Rightarrow \ Q(a,c) = g(Q(a,b), Q(b,c)) \,.$$

The conditions imposed upon g again ensure that g-isostochastic transitivity generalizes transitivity of complete relations. Note that for a given mapping g, the property of g-isostochastic transitivity is much more restrictive than the property of g-stochastic transitivity.

***FG*-transitivity** The framework of FG-transitivity, developed by Switalski (2001, 2003), formally generalizes g-stochastic transitivity in the sense that $Q(a, c)$ is bounded both from below and above by $[1/2, 1]^2 \to [0, 1]$ mappings.

Definition 2.4. Let F and G be two $[1/2, 1]^2 \to [0, 1]$ mappings such that $F(1/2, 1/2) \leq 1/2 \leq G(1/2, 1/2)$, and $G(1/2, 1) = G(1, 1/2) = G(1, 1) = 1$ and $F \leq G$. A reciprocal relation Q on X is called FG-transitive if for any $(a, b, c) \in X^3$ it holds that

$$\begin{gathered}(Q(a,b) \geq 1/2 \wedge Q(b,c) \geq 1/2) \\ \Downarrow \\ F(Q(a,b), Q(b,c)) \leq Q(a,c) \leq G(Q(a,b), Q(b,c)) \,.\end{gathered}$$

Cycle-transitivity For a reciprocal relation Q, we define for all $(a,b,c) \in X^3$ the following quantities, see De Baets et al. (2006a):

$$\begin{aligned}\alpha_{abc} &= \min(Q(a,b), Q(b,c), Q(c,a))\,,\\ \beta_{abc} &= \text{median}(Q(a,b), Q(b,c), Q(c,a))\,,\\ \gamma_{abc} &= \max(Q(a,b), Q(b,c), Q(c,a))\,.\end{aligned}$$

Let us also denote $\Delta = \{(x,y,z) \in [0,1]^3 \mid x \le y \le z\}$. A function $U : \Delta \to \mathbb{R}$ is called an upper bound function if it satisfies:

(i) $U(0,0,1) \ge 0$ and $U(0,1,1) \ge 1$;

(ii) for any $(\alpha,\beta,\gamma) \in \Delta$:

$$U(\alpha,\beta,\gamma) + U(1-\gamma, 1-\beta, 1-\alpha) \ge 1\,. \tag{4}$$

The function $L : \Delta \to \mathbb{R}$ defined by $L(\alpha,\beta,\gamma) = 1 - U(1-\gamma,1-\beta,1-\alpha)$ is called the *dual lower bound function* of a given upper bound function U. Inequality (4) then simply expresses that $L \le U$. Condition (i) again guarantees that cycle-transitivity generalizes transitivity of complete relations.

Definition 2.5. A reciprocal relation Q on X is called cycle-transitive w.r.t. an upper bound function U if for any $(a,b,c) \in X^3$ it holds that

$$L(\alpha_{abc},\beta_{abc},\gamma_{abc}) \le \alpha_{abc} + \beta_{abc} + \gamma_{abc} - 1 \le U(\alpha_{abc},\beta_{abc},\gamma_{abc})\,, \tag{5}$$

where L is the dual lower bound function of U.

Due to the built-in duality, it holds that if (5) is true for some (a,b,c), then this is also the case for any permutation of (a,b,c). In practice, it is therefore sufficient to check (5) for a single permutation of any $(a,b,c) \in X^3$. Alternatively, due to the same duality, it is also sufficient to verify the right-hand inequality (or equivalently, the left-hand inequality) for two permutations of any $(a,b,c) \in X^3$ (not being cyclic permutations of one another), e.g. (a,b,c) and (c,b,a). Hence, (5) can be replaced by

$$\alpha_{abc} + \beta_{abc} + \gamma_{abc} - 1 \le U(\alpha_{abc},\beta_{abc},\gamma_{abc})\,.$$

Note that a value of $U(\alpha,\beta,\gamma)$ equal to 2 is used to express that for the given values there is no restriction at all (as $\alpha+\beta+\gamma-1$ is always bounded by 2).

Two upper bound functions U_1 and U_2 are called *equivalent* if for any $(\alpha,\beta,\gamma) \in \Delta$ it holds that $\alpha+\beta+\gamma-1 \le U_1(\alpha,\beta,\gamma)$ is equivalent to $\alpha+\beta+\gamma-1 \le U_2(\alpha,\beta,\gamma)$.

If it happens that in (4) the equality holds for all $(\alpha,\beta,\gamma)\in\Delta$, then the upper bound function U is said to be *self-dual*, since in that case it coincides with its dual lower bound function L. Consequently, also (5) and (2.2) can only hold with equality. Furthermore, it then holds that $U(0,0,1)=0$ and $U(0,1,1)=1$.

Although C-transitivity is not intended to be applied to reciprocal relations, it can be cast quite nicely into the cycle-transitivity framework of De Baets et al. (2006a).

Proposition 2.6. *Let C be a commutative conjunctor such that $C\leq T_{\mathbf{M}}$. A reciprocal relation Q on X is C-transitive if and only if it is cycle-transitive w.r.t. the upper bound function U_f defined by*

$$U_C(\alpha,\beta,\gamma)=\min(\alpha+\beta-C(\alpha,\beta),\beta+\gamma-C(\beta,\gamma),\gamma+\alpha-C(\gamma,\alpha))\,.$$

Moreover, if C is 1-Lipschitz, then U_C is given by

$$U_C(\alpha,\beta,\gamma)=\alpha+\beta-C(\alpha,\beta)\,.$$

Consider the three basic t-norms (copulas) $T_{\mathbf{M}}$, $T_{\mathbf{P}}$ and $T_{\mathbf{L}}$:

(i) For $C=T_{\mathbf{M}}$, we immediately obtain as upper bound function the median (the simplest self-dual upper bound function):

$$U_{T_{\mathbf{M}}}(\alpha,\beta,\gamma)=\beta\,.$$

(ii) For $C=T_{\mathbf{P}}$, we find

$$U_{T_{\mathbf{P}}}(\alpha,\beta,\gamma)=\alpha+\beta-\alpha\beta\,.$$

(iii) For $C=T_{\mathbf{L}}$, we obtain

$$U_{T_{\mathbf{L}}}(\alpha,\beta,\gamma)=\begin{cases}\alpha+\beta & ,\ \text{if } \alpha+\beta<1\,,\\ 1 & ,\ \text{if } \alpha+\beta\geq 1\,.\end{cases}$$

An equivalent upper bound function is given by $U'_{T_{\mathbf{L}}}(\alpha,\beta,\gamma)=1$.

Cycle-transitivity also incorporates stochastic transitivity, although the latter fits more naturally in the FG-transitivity framework; in particular, isostochastic transitivity corresponds to cycle-transitivity w.r.t. particular self-dual upper bound functions (De Baets et al. (2006a)). We have shown that cycle-transitivity and FG-transitivity frameworks cannot easily be translated into one another, which underlines that these are two essentially different frameworks (De Baets and De Meyer (2005b)).

One particular form of stochastic transitivity deserves our attention. A probabilistic relation Q on X is called *partially stochastic transitive* (Fishburn (1973)) if for any $(a,b,c) \in X^3$ it holds that

$$(Q(a,b) > 1/2 \wedge Q(b,c) > 1/2) \Rightarrow Q(a,c) \geq \min(Q(a,b), Q(b,c)) .$$

Clearly, it is a slight weakening of moderate stochastic transitivity. Interestingly, also this type of transitivity can be expressed elegantly in the cycle-transitivity framework (De Meyer et al. (2007)) by means of a simple upper bound function.

Proposition 2.7. *Cycle-transitivity w.r.t. the upper bound function U_{ps} defined by*

$$U_{ps}(\alpha, \beta, \gamma) = \gamma$$

is equivalent to partial stochastic transitivity.

3 Similarity of fuzzy sets

3.1 Basic notions

Recall that an *equivalence relation* E on X is a reflexive, symmetric and transitive relation on X and that there exists a one-to-one correspondence between equivalence relations on X and partitions of X. In fuzzy set theory, the counterpart of an equivalence relation is a T-*equivalence*: given a t-norm T, a T-equivalence E on X is a fuzzy relation on X that is reflexive ($E(x,x) = 1$), symmetric ($E(x,y) = E(y,x)$) and T-transitive. A T-equivalence is called a T-equality if $E(x,y)$ implies $x = y$.

For the prototypical t-norms, it is interesting to note that (see e.g. De Baets and Mesiar (1997, 2002)):

(i) A fuzzy relation E on X is a $T_{\mathbf{L}}$-equivalence if and only if $d = 1 - E$ is a pseudo-metric on X.

(ii) A fuzzy relation E on X is a $T_{\mathbf{P}}$-equivalence if and only if $d = -\log E$ is a pseudo-metric on X.

(iii) A fuzzy relation E on X is a $T_{\mathbf{M}}$-equivalence if and only if $d = 1 - E$ is a pseudo-ultra-metric on X. Another interesting characterization is that a fuzzy relation E on X is a $T_{\mathbf{M}}$-equivalence if and only if for any $\alpha \in [0,1]$ its α-cut $E_\alpha = \{(x,y) \in X^2 \mid E(x,y) \geq \alpha\}$ is an equivalence relation on X. The equivalence classes of E_α become smaller for increasing α leading to the concept of a partition tree (see e.g. De Meyer et al. (2004)).

3.2 A logical approach

To any left-continuous t-norm T, there corresponds a residual implicator $I_T : [0,1]^2 \to [0,1]$ defined by

$$I_T(x,y) = \sup\{z \in [0,1] \mid T(x,z) \leq y\}\,,$$

which can be considered as a generalization of Boolean implication. Note that $I_T(x,y) = 1$ if and only if $x \leq y$. In case $y < x$, one gets for the prototypical t-norms: $I_{\mathbf{M}}(x,y) = y$, $I_{\mathbf{P}}(x,y) = y/x$ and $I_{\mathbf{L}}(x,y) = \min(1-x+y,1)$. An essential property of the residual implicator of a left-continuous t-norm is related to the classical syllogism:

$$T(I_T(x,y), I_T(y,z)) \leq I_T(x,z))\,,$$

for any $(x,y,z) \in [0,1]^3$. The residual implicator is the main constituent of the biresidual operator $\mathcal{E}_T : [0,1]^2 \to [0,1]$ defined by

$$\mathcal{E}_T(x,y) = \min(I_T(x,y), I_T(y,x)) = I_T(\max(x,y), \min(x,y))\,,$$

which can be considered as a generalization of Boolean equivalence. Note that $\mathcal{E}_T(x,y) = 1$ if and only if $x = y$. In case $x \neq y$, one gets for the prototypical t-norms: $\mathcal{E}_{\mathbf{M}}(x,y) = \min(x,y)$, $\mathcal{E}_{\mathbf{P}}(x,y) = \min(x,y)/\max(x,y)$ and $\mathcal{E}_{\mathbf{L}}(x,y) = 1 - |x-y|$.

Of particular importance in this discussion is the fact that $\mathcal{E}_T$ is a T-equality on $[0,1]$. The biresidual operator obviously serves as a means for measuring equality of membership degrees. Any T-equality E on $[0,1]$ can be extended in a natural way to $\mathcal{F}(X)$, the class of fuzzy sets in X:

$$E'(A,B) = \inf_{x \in X} E(A(x), B(x))\,.$$

It then holds that E' is a T-equality on $\mathcal{F}(X)$ if and only if E is a T-equality on $[0,1]$. Starting from $\mathcal{E}_T$ we obtain the T-equality E^T. A second way of defining a T-equality on $\mathcal{F}(X)$ is by defining

$$E_T(A,B) = T(\inf_{x \in X} I_T(A(x), B(x)), \inf_{x \in X} I_T(B(x), A(x)))\,.$$

The underlying idea is that in order to measure equality of two (fuzzy) sets A and B, one should both measure inclusion of A in B, and of B in A. Note that in general $E_T \subseteq E^T$, while $E_{\mathbf{M}} = E^{\mathbf{M}}$. These T-equivalences can be used as a starting point for building metrics on $\mathcal{F}(X)$. The above ways of measuring equality of fuzzy sets are very strict in the sense that the "worst" element decides upon the value.

Without going into detail, it is worth mentioning that there exist an appropriate notion of fuzzy partition, called T-partition (De Baets and Mesiar (1998)), so that there exists a one-to-one correspondence between T-equalities on X and T-partitions of X (De Baets and Mesiar (2002)).

3.3 A cardinal approach

Classical cardinality-based similarity measures A common recipe for comparing objects is to select an appropriate set of features and to construct for each object a binary vector encoding the presence (1) or absence (0) of each of these features. Such a binary vector can be formally identified with the corresponding set of present features. The degree of similarity of two objects is then often expressed in terms of the cardinalities of the latter sets. We focus our attention on a family of $[0,1]$-valued similarity measures that are rational expressions in the cardinalities of the sets involved, see De Baets et al. (2001):

$$S(A,B) = \frac{x\,\alpha_{A,B} + t\,\omega_{A,B} + y\,\delta_{A,B} + z\,\nu_{A,B}}{x'\,\alpha_{A,B} + t'\,\omega_{A,B} + y'\,\delta_{A,B} + z'\,\nu_{A,B}}\,,$$

with $A, B \in \mathcal{P}(X)$ (the powerset of a finite universe X),

$$\begin{aligned}
\alpha_{A,B} &= \min(|A \setminus B|, |B \setminus A|)\,,\\
\omega_{A,B} &= \max(|A \setminus B|, |B \setminus A|)\,,\\
\delta_{A,B} &= |A \cap B|\,,\\
\nu_{A,B} &= |(A \cup B)^c|\,,
\end{aligned}$$

and $x, t, y, z, x', t', y', z' \in \{0,1\}$. Note that these similarity measures are symmetric, *i.e.* $S(A,B) = S(B,A)$ for any $A, B \in \mathcal{P}(X)$.

Reflexive similarity measures, *i.e.* $S(A,A) = 1$ for any $A \in \mathcal{P}(X)$, are characterized by $y = y'$ and $z = z'$. We restrict our attention to the (still large) subfamily obtained by putting also $t = x$ and $t' = x'$ (De Baets and De Meyer (2005a); De Baets et al. (to appear)), *i.e.*

$$S(A,B) = \frac{x\,\triangle_{A,B} + y\,\delta_{A,B} + z\,\nu_{A,B}}{x'\,\triangle_{A,B} + y\,\delta_{A,B} + z\,\nu_{A,B}}\,, \tag{6}$$

with $\triangle_{A,B} = |A \triangle B| = |A \setminus B| + |B \setminus A|$. On the other hand, we allow more freedom by letting the parameters x, y, z and x' take positive real values. Note that these parameters can always be scaled to the unit interval by dividing both numerator and denominator of (6) by the greatest among the parameters. In order to guarantee that $S(A,B) \in [0,1]$, we need to impose the restriction $0 \le x \le x'$. Since the case $x = x'$ leads to trivial measures taking value 1 only, we consider from here on $0 \le x < x'$. The similarity measures gathered in Table 1 all belong to family (6); the corresponding parameter values are indicated in the table.

The $T_{\mathbf{L}}$- or $T_{\mathbf{P}}$-transitive members of family (6) are characterized in the following proposition (De Baets et al. (to appear)).

Measure	expression	x	x'	y	z	T
Jaccard (1908)	$\frac{\lvert A\cap B\rvert}{\lvert A\cup B\rvert}$	0	1	0	1	$T_{\mathbf{L}}$
Simple Matching (Sokal and Michener, 1958)	$1-\frac{\lvert A\triangle B\rvert}{n}$	0	1	1	1	$T_{\mathbf{L}}$
Dice (1945)	$\frac{2\lvert A\cap B\rvert}{\lvert A\triangle B\rvert+2\lvert A\cap B\rvert}$	0	1	2	0	–
Rogers and Tanimoto (1960)	$\frac{n-\lvert A\triangle B\rvert}{n+\lvert A\triangle B\rvert}$	0	2	1	1	$T_{\mathbf{L}}$
Sneath and Sokal (1973)	$\frac{\lvert A\cap B\rvert}{\lvert A\cap B\rvert+2\lvert A\triangle B\rvert}$	0	2	1	0	$T_{\mathbf{L}}$
Sneath and Sokal (1973)	$1-\frac{\lvert A\triangle B\rvert}{2n-\lvert A\triangle B\rvert}$	0	1	2	2	–

Table 1. Some well-known cardinality-based similarity measures.

Proposition 3.1.
(i) The $T_{\mathbf{L}}$-transitive members of family (6) are characterized by the necessary and sufficient condition $x' \geq \max(y, z)$.
(ii) The $T_{\mathbf{P}}$-transitive members of family (6) are characterized by the necessary and sufficient condition $x\,x' \geq \max(y^2, z^2)$.

Fuzzy cardinality-based similarity measures Often, the presence or absence of a feature is not clear-cut and is rather a matter of degree. Hence, if instead of binary vectors we have to compare vectors with components in the real unit interval $[0, 1]$ (the higher the number, the more the feature is present), the need arises to generalize the aforementioned similarity measures. In fact, in the same way as binary vectors can be identified with ordinary subsets of a finite universe X, vectors with components in $[0, 1]$ can be identified with fuzzy sets in X.

In order to generalize cardinality-based similarity measure to fuzzy sets, we clearly need fuzzification rules that define the cardinality of a fuzzy set and translate the classical set-theoretic operations to fuzzy sets. As to the first, we stick to the following simple way of defining the cardinality of a fuzzy set, also known as the sigma-count of A (Zadeh (1965)): $|A| = \sum_{x\in X} A(x)$. As to the second, we define the intersection of two fuzzy sets A and B in X in a pointwise manner by $A \cap B(x) = C(A(x), B(x))$, for any $x \in X$, where C is a commutative conjunctor. In De Baets et al. (to appear), we have argued that commutative quasi-copulas are the most appropriate conjunctors for our purpose. Commutative quasi-copulas not only allow to introduce set-theoretic operations on fuzzy sets, such as $A \setminus B(x) = A(x) - C(A(x), B(x))$ and $A \triangle B(x) = A(x) + B(x) - 2C(A(x), B(x))$, they also preserve classical identities on cardinalities, such as $|A \setminus B| = |A| - |A \cap B|$

and $|A \triangle B| = |A \setminus B| + |B \setminus A| = |A| + |B| - 2|A \cap B|$. These identities allow to rewrite and fuzzify family (6) as

$$S(A,B) = \frac{x(a+b-2u) + yu + z(n-a-b+u)}{x'(a+b-2u) + yu + z(n-a-b+u)}\,, \tag{7}$$

with $a = |A|$, $b = |B|$ and $u = |A \cap B|$.

Bell-inequalities and preservation of transitivity Studying the transitivity of (fuzzy) cardinality-based similarity measures inevitably leads to the verification of inequalities on (fuzzy) cardinalities. We have established several powerful meta-theorems that provide an efficient and intelligent way of verifying whether a classical inequality on cardinalities carries over to fuzzy cardinalities (De Baets et al. (2006b)). These meta-theorems state that certain classical inequalities are preserved under fuzzification when modelling fuzzy set intersection by means of a commutative conjunctor that fulfills a number of Bell-type inequalities.

In Janssens et al. (2004a), we introduced the classical Bell inequalities in the context of fuzzy probability calculus and proved that the following Bell-type inequalities for commutative conjunctors are necessary and sufficient conditions for the corresponding Bell-type inequalities for fuzzy probabilities to hold. The Bell-type inequalities for a commutative conjunctor C read as follows:

$$\begin{aligned}
&B_1 : T_{\mathbf{L}}(p,q) \le C(p,q) \le T_{\mathbf{M}}(p,q)\\
&B_2 : 0 \le p - C(p,q) - C(p,r) + C(q,r)\\
&B_3 : p + q + r - C(p,q) - C(p,r) - C(q,r) \le 1
\end{aligned}$$

for any $p,q,r \in [0,1]$. Inequality B_2 is fulfilled for any commutative quasi-copula, while inequality B_3 only holds for certain t-norms Janssens et al. (2004b), including the members of the Frank t-norm family $T^{\mathbf{F}}_{\lambda}$ with $\lambda \le 9+4\sqrt{5}$ (Pykacz and D'Hooghe (2001)). Also note that inequality B_1 follows from inequality B_2.

Theorem 3.2. *(De Baets et al. (2006b)) Consider a commutative conjunctor I that satisfies Bell inequalities B_2 and B_3. If for any ordinary subsets A, B and C of an arbitrary finite universe X it holds that*

$$\mathcal{H}(|A|,|B|,|C|,|A\cap B|,|A\cap C|,|B\cap C|,|X|) \ge 0\,,$$

where $\mathcal{H}$ denotes a continuous function which is homogeneous in its arguments, then it also holds for any fuzzy sets in an arbitrary finite universe Y.

If the function $\mathcal{H}$ does not depend explicitly upon $|X|$, then Bell inequality B_3 can be omitted. This meta-theorem allows us to identify conditions on the parameters of the members of family (7) leading to $T_{\mathbf{L}}$-transitive or $T_{\mathbf{P}}$-transitive fuzzy similarity measures. As our fuzzification is based on a commutative quasi-copula C, condition B_2 holds by default. The following proposition then is an immediate application (De Baets et al. (2006b)).

Proposition 3.3.
- *(i) Consider a commutative quasi-copula C that satisfies B_3. The $T_{\mathbf{L}}$-transitive members of family (7) are characterized by $x' \geq \max(y, z)$.*
- *(ii) The $T_{\mathbf{L}}$-transitive members of family (7) with $z = 0$ are characterized by $x' \geq y$.*
- *(iii) Consider a commutative quasi-copula C that satisfies B_3. The $T_{\mathbf{P}}$-transitive members of family (7) are characterized by $x\,x' \geq \max(y^2, z^2)$.*
- *(iv) The $T_{\mathbf{P}}$-transitive members of family (7) with $z = 0$ are characterized by $xx' \geq y^2$.*

However, as our meta-theorem is very general, it does not necessarily always provide the strongest results. For instance, tedious and lengthy direct proofs allow to eliminate condition B_3 from the previous theorem, leading to the following general result (De Baets et al. (2006b)).

Proposition 3.4. *Consider a commutative quasi-copula C.*
- *(i) The $T_{\mathbf{L}}$-transitive members of family (7) are characterized by the necessary and sufficient condition $x' \geq \max(y, z)$.*
- *(ii) The $T_{\mathbf{P}}$-transitive members of family (7) are characterized by the necessary and sufficient condition $x\,x' \geq \max(y^2, z^2)$.*

4 Comparison of random variables

4.1 Dice-transitivity

Consider three dice A, B and C which, instead of the usual numbers, carry the following integers on their faces:

$$A = \{1, 3, 4, 15, 16, 17\}, \quad B = \{2, 10, 11, 12, 13, 14\}, \quad C = \{5, 6, 7, 8, 9, 18\}\,.$$

Denoting by $\mathcal{P}(X, Y)$ the probability that dice X wins from dice Y, we have $\mathcal{P}(A, B) = 20/36$, $\mathcal{P}(B, C) = 25/36$ and $\mathcal{P}(C, A) = 21/36$. It is natural to say that dice X is strictly preferred to dice Y if $\mathcal{P}(X, Y) > 1/2$, which reflects that dice X wins from dice Y in the long run (or that X statistically wins from Y, denoted $X >_s Y$). Note that $\mathcal{P}(Y, X) = 1 - \mathcal{P}(X, Y)$ which implies that the relation $>_s$ is asymmetric. In the above example, it holds that $A >_s B$, $B >_s C$ and $C >_s A$: the relation $>_s$ is not transitive and forms a cycle. In other words, if we interpret the probabilities $\mathcal{P}(X, Y)$ as constituents of a reciprocal relation on the set of alternatives $\{A, B, C\}$, then this reciprocal relation is even not weakly stochastic transitive.

This example can be generalized as follows: we allow the dice to possess any number of faces (whether or not this can be materialized) and allow identical numbers on the faces of a single or multiple dice. In other words, a generalized dice can be identified with a multiset of integers. Given a collection of m such generalized dice, we can still build a reciprocal relation Q containing the *winning probabilities* for each pair of dice (De Schuymer et al. (2003)). For any two such dice A and B, we define

$$Q(A,B) = \mathcal{P}\{A \text{ wins from } B\} + \frac{1}{2}\mathcal{P}\{A \text{ and } B \text{ end in a tie}\}\,.$$

The dice or integer multisets may be identified with independent discrete random variables that are uniformly distributed on these multisets (i.e. the probability of an integer is proportional to its number of occurences); the reciprocal relation Q may be regarded as a quantitative description of the pairwise comparison of these random variables.

In the characterization of the transitivity of this reciprocal relation, a type of cycle-transitivity, which can neither be seen as a type of C-transitivity, nor as a type of FG-transitivity, has proven to play a predominant role. For obvious reasons, this new type of transitivity has been called dice-transitivity.

Definition 4.1. Cycle-transitivity w.r.t. the upper bound function U_D defined by

$$U_D(\alpha,\beta,\gamma) = \beta + \gamma - \beta\gamma\,,$$

is called *dice-transitivity*.

Dice-transitivity is closely related to $T_{\mathbf{P}}$-transitivity. However, it uses the quantities β and γ instead of the quantities α and β, and is therefore less restrictive. Dice-transitivity can be situated between $T_{\mathbf{L}}$-transitivity and $T_{\mathbf{P}}$-transitivity, and also between $T_{\mathbf{L}}$-transitivity and moderate stochastic transitivity.

Proposition 4.2. *(De Schuymer et al. (2003)) The reciprocal relation generated by a collection of generalized dice is dice-transitive.*

4.2 A method for comparing random variables

Many methods can be established for the comparison of the components (random variables, r.v.) of a random vector $(X_1, \ldots, X_n)$, as there exist many ways to extract useful information from the joint cumulative distribution function (c.d.f.) $F_{X_1,\ldots,X_n}$ that characterizes the random vector. A first simplification consists in comparing the r.v. two by two. It means that a method for comparing r.v. should only use the information contained in the bivariate c.d.f. F_{X_i,X_j}. Therefore, one can very well ignore the existence of a multivariate c.d.f. and just describe mutual dependencies between the r.v. by means of the bivariate c.d.f. Of course one should be aware that not all choices of bivariate c.d.f. are compatible with a multivariate c.d.f. The problem of characterizing those ensembles of bivariate c.d.f. that can be identified with the marginal bivariate c.d.f. of a single multivariate c.d.f., is known as the *compatibility problem* (Nelsen (1998)).

A second simplifying step often made is to bypass the information contained in the bivariate c.d.f. to devise a comparison method that entirely relies on the one-dimensional marginal c.d.f. In this case there is even not a compatibility problem, as for any set of univariate c.d.f. F_{X_i}, the product $F_{X_1}F_{X_2}\cdots F_{X_n}$ is a valid joint c.d.f., namely the one expressing the independence of the r.v. There are many ways to compare one-dimensional c.d.f., and by far the simplest one is the method that builds a partial order on the set of r.v. using the principle of first order stochastic dominance (Levy (1998)). It states that a r.v. X is weakly preferred to a r.v. Y if for all $u \in \mathbb{R}$ it holds that $F_X(u) \leq F_Y(u)$. At the extreme end of the chain of simplifications, are the methods that compare r.v. by means of a characteristic or a function of some characteristics derived from the one-dimensional marginal c.d.f. The simplest example is the weak order induced by the expected values of the r.v.

Proceeding along the line of thought of the previous section, a random vector $(X_1, X_2, \ldots, X_m)$ generates a reciprocal relation by means of the following recipe.

Definition 4.3. Given a random vector $(X_1, X_2, \ldots, X_m)$, the binary relation Q defined by

$$Q(X_i, X_j) = \mathcal{P}\{X_i > X_j\} + \frac{1}{2}\,\mathcal{P}\{X_i = X_j\}$$

is a reciprocal relation.

For two discrete r.v. X_i and X_j, $Q(X_i, X_j)$ can be computed as

$$Q(X_i, X_j) = \sum_{k>l} p_{X_i,X_j}(k,l) + \frac{1}{2}\sum_{k} p_{X_i,X_j}(k,k)\,,$$

with p_{X_i,X_j} the joint probability mass function (p.m.f.) of (X_i, X_j). For two continuous r.v. X_i and X_j, $Q(X_i, X_j)$ can be computed as:

$$Q(X_i, X_j) = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{x} f_{X_i,X_j}(x,y)\,dy\,,$$

with f_{X_i,X_j} the joint probability density function (p.d.f.) of (X_i, X_j).

For this pairwise comparison, one needs the two-dimensional marginal distributions. Sklar's theorem (Sklar (1959); Nelsen (1998)) tells us that if a joint cumulative distribution function F_{X_i,X_j} has marginals F_{X_i} and F_{X_j}, then there exists a copula C_{ij} such that for all x, y:

$$F_{X_i,X_j}(x, y) = C_{ij}(F_{X_i}(x), F_{X_j}(y)) \,.$$

If X_i and X_j are continuous, then C_{ij} is unique; otherwise, C_{ij} is uniquely determined on $\mathrm{Ran}(F_{X_i}) \times \mathrm{Ran}(F_{X_j})$.

As the above comparison method takes into account the bivariate marginal c.d.f. it takes into account the dependence of the components of the random vector. The information contained in the reciprocal relation is therefore much richer than if, for instance, we would have based the comparison of X_i and X_j solely on their expected values. Despite the fact that the dependence structure is entirely captured by the multivariate c.d.f., the pairwise comparison is only apt to take into account pairwise dependence, as only bivariate c.d.f. are involved. Indeed, the bivariate c.d.f. do not fully disclose the dependence structure; the r.v. may even be pairwise independent while not mutually independent.

Since the copulas C_{ij} that couple the univariate marginal c.d.f. into the bivariate marginal c.d.f. can be different from another, the analysis of the reciprocal relation and in particular the identification of its transitivity properties appear rather cumbersome. It is nonetheless possible to state in general, without making any assumptions on the bivariate c.d.f., that the probabilistic relation Q generated by an arbitrary random vector always shows some minimal form of transitivity (De Baets and De Meyer (2008)).

Proposition 4.4. *The reciprocal relation Q generated by a random vector is $T_{\mathbf{L}}$-transitive.*

4.3 Artificial coupling of random variables

Our further interest is to study the situation where abstraction is made that the r.v. are components of a random vector, and all bivariate c.d.f. are enforced to depend in the same way upon the univariate c.d.f., in other words, we consider the situation of all copulas being the same, realizing that this might not be possible at all. In fact, this simplification is equivalent to considering instead of a random vector, a collection of r.v. and to artificially compare them, all in the same manner and based upon a same copula. The pairwise comparison then relies upon the knowledge of the one-dimensional marginal c.d.f. solely, as is the case in stochastic dominance methods. Our comparison method, however, is not equivalent to any known kind of stochastic dominance, but should rather be regarded as a graded variant of it (see also De Baets and De Meyer (2007)).

The case $C = T_{\mathbf{P}}$ generalizes Proposition 4.2, and applies in particular to a collection of independent r.v. where all copulas effectively equal $T_{\mathbf{P}}$.

Proposition 4.5. *(De Schuymer et al. (2003, 2005)) The reciprocal relation Q generated by a collection of r.v. pairwisely coupled by $T_{\mathbf{P}}$ is dice-transitive,* i.e. *it is cycle-transitive w.r.t. the upper bound function given by $U_D(\alpha, \beta, \gamma) = \beta + \gamma - \beta\gamma$.*

Next, we discuss the case when using one of the extreme copulas to artificially couple the r.v. In case $C = T_{\mathbf{M}}$, the r.v. are coupled comonotonically. Note that this case is possible in reality.

Proposition 4.6. *(De Schuymer et al. (2007); De Meyer et al. (2007)) The reciprocal relation Q generated by a collection of r.v. pairwisely coupled by $T_{\mathbf{M}}$ is cycle-transitive w.r.t. to the upper bound function U given by $U(\alpha, \beta, \gamma) = \min(\beta+\gamma, 1)$. Cycle-transitivity w.r.t. the upper bound function U is equivalent to $T_{\mathbf{L}}$-transitivity.*

In case $C = T_{\mathbf{L}}$, the r.v. are coupled countermonotonically. This assumption can never represent a true dependence structure for more than two r.v., due to the compatibility problem.

Proposition 4.7. *(De Schuymer et al. (2007); De Meyer et al. (2007)) The reciprocal relation Q generated by a collection of r.v. pairwisely coupled by $T_{\mathbf{L}}$ is partially stochastic transitive,* i.e. *it is cycle-transitive w.r.t. to the upper bound function defined by $U_{ps}(\alpha, \beta, \gamma) = \max(\beta, \gamma) = \gamma$.*

The proofs of these propositions were first given for discrete uniformly distributed r.v. (De Schuymer et al. (2003, 2007)). It allowed for an interpretation of the values $Q(X_i, X_j)$ as winning probabilities in a hypothetical dice game, or equivalently, as a method for the pairwise comparison of ordered lists of numbers. Subsequently, we have shown that as far as transitivity is concerned, this situation is generic and therefore characterizes the type of transitivity observed in general (De Meyer et al. (2007); De Schuymer et al. (2005)).

The above results can be seen as particular cases of a more general result (see De Baets and De Meyer (2008)).

Proposition 4.8. *Let C be a commutative copula such that for any $n > 1$ and for any $0 \leq x_1 \leq \cdots \leq x_n \leq 1$ and $0 \leq y_1 \leq \cdots \leq y_n \leq 1$, it holds that*

$$\sum_i C(x_i, y_i) - \sum_i C(x_{n-2i}, y_{n-2i-1}) - \sum_i C(x_{n-2i-1}, y_{n-2i})$$
$$\leq C\Bigg(x_n + \sum_i C(x_{n-2i-2}, y_{n-2i-1}) - \sum_i C(x_{n-2i}, y_{n-2i-1}),$$
$$y_n + \sum_i C(x_{n-2i-1}, y_{n-2i-2}) - \sum_i C(x_{n-2i-1}, y_{n-2i})\Bigg)\,, \quad (8)$$

where the sums extend over all integer values that lead to meaningful indices of x and y. Then the reciprocal relation Q generated by a collection of random variables pairwisely coupled by C is cycle-transitive w.r.t. to the upper bound function U^C defined by:

$$U^C(\alpha, \beta, \gamma) = \max(\beta + C(1-\beta, \gamma), \gamma + C(\beta, 1-\gamma))\,.$$

Inequality (8) is called the *twisted staircase condition* and appears to be quite complicated. Although its origin is well understood (see De Baets and De Meyer (2008)), it is not yet clear for which commutative copulas it holds. We strongly conjecture that it holds for all Frank copulas.

4.4 Comparison of special independent random variables

Dice-transitivity is the generic type of transitivity shared by the reciprocal relations generated by a collection of independent r.v. If one considers independent r.v. with densities all belonging to one of the one-parameter families in Table 2, the corresponding reciprocal relation shows the corresponding type of cycle-transitivity listed in Table 4.4 (De Schuymer et al. (2005)).

Note that all upper bound functions in Table 3 are self-dual. More striking is that the two families of power-law distributions (one-parameter subfamilies of the two-parameter Beta and Pareto families) and the family of Gumbel distributions, all yield the same type of transitivity as exponential distributions, namely cycle-transitivity w.r.t. the self-dual upper bound function U_E defined by:

$$U_E(\alpha, \beta, \gamma) = \alpha\beta + \alpha\gamma + \beta\gamma - 2\alpha\beta\gamma\,.$$

Cycle-transitivity w.r.t. U_E can also be expressed as

$$\alpha_{abc}\beta_{abc}\gamma_{abc} = (1-\alpha_{abc})(1-\beta_{abc})(1-\gamma_{abc})\,,$$

Table 2. Parametric families of continuous distributions.

Name	Density function $f(x)$		
Exponential	$\lambda e^{-\lambda x}$	$\lambda > 0$	$x \in [0, \infty[$
Beta	$\lambda x^{(\lambda-1)}$	$\lambda > 0$	$x \in [0, 1]$
Pareto	$\lambda x^{-(\lambda+1)}$	$\lambda > 0$	$x \in [1, \infty[$
Gumbel	$\mu e^{-\mu(x-\lambda)} e^{-e^{-\mu(x-\lambda)}}$	$\lambda \in \mathbb{R}$, $\mu > 0$	$x \in]-\infty, \infty[$
Uniform	$1/a$	$\lambda \in \mathbb{R}$, $a > 0$	$x \in [\lambda, \lambda + a]$
Laplace	$(e^{-\lvert x-\lambda\rvert/\mu)})/(2\mu)$	$\lambda \in \mathbb{R}$, $\mu > 0$	$x \in]-\infty, \infty[$
Normal	$(e^{-(x-\lambda)^2/2\sigma^2})/\sqrt{2\pi\sigma^2}$	$\lambda \in \mathbb{R}, \sigma > 0$	$x \in]-\infty, \infty[$

which is equivalent to the notion of multiplicative transitivity of Tanino (1988). A reciprocal relation Q on X is called *multiplicatively transitive* if for any $(a, b, c) \in X^3$ it holds that

$$\frac{Q(a,c)}{Q(c,a)} = \frac{Q(a,b)}{Q(b,a)} \cdot \frac{Q(b,c)}{Q(c,b)} \,.$$

In the cases of the unimodal uniform, Gumbel, Laplace and normal distributions we have fixed one of the two parameters in order to restrict the family to a one-parameter subfamily, mainly because with two free parameters, the formulae become utmost cumbersome. The one exception is the two-dimensional family of normal distributions. In De Schuymer et al. (2005), we have shown that the corresponding reciprocal relation is in that case moderately stochastic transitive.

5 Conclusion

We have introduced the reader to two relational frameworks and the wide variety of transitivity notions available in them. The presentation was rather dense and more information can be found following the many literature pointers given.

Anticipating on future work, in particular on applications, we can identify two important directions. The first direction concerns the use of fuzzy similarity measures. Moser (2006) has shown recently that the T-equality E^T, with $T = T_{\mathbf{P}}$ or $T = T_{\mathbf{L}}$, is positive semi-definite. We are currently tackling the same question for the fuzzy cardinality-based similarity measures. Results of this type allow to bridge the gap between the fuzzy set

Table 3. Cycle-transitivity for the continuous distributions in Table 1.

<table>
<tr><th>Name</th><th>Upper bound function $U(\alpha,\beta,\gamma)$</th></tr>
<tr><td>Exponential
Beta
Pareto
Gumbel</td><td>$\alpha\beta+\alpha\gamma+\beta\gamma-2\alpha\beta\gamma$</td></tr>
<tr><td>Uniform</td><td>$\begin{cases} \beta+\gamma-1+\frac{1}{2}[\max(\sqrt{2(1-\beta)}+\sqrt{2(1-\gamma)}-1,0)]^2 & ,\beta\geq 1/2 \\ \alpha+\beta-\frac{1}{2}[\max(\sqrt{2\alpha}+\sqrt{2\beta}-1,0)]^2 & ,\beta<1/2 \end{cases}$</td></tr>
<tr><td>Laplace</td><td>$\begin{cases} \beta+\gamma-1+f^{-1}(f(1-\beta)+f(1-\gamma)) & ,\beta\geq 1/2 \\ \alpha+\beta-f^{-1}(f(\alpha)+f(\beta)) & ,\beta<1/2 \end{cases}$
with $f^{-1}(x)=\frac{1}{2}\left(1+\frac{x}{2}\right)e^{-x}$</td></tr>
<tr><td>Normal</td><td>$\begin{cases} \beta+\gamma-1+\Phi(\Phi^{-1}(1-\beta)+\Phi(1-\gamma)) & ,\beta\geq 1/2 \\ \alpha+\beta-\Phi(\Phi^{-1}(\alpha)+\Phi^{-1}(\beta)) & ,\beta<1/2 \end{cases}$
with $\Phi(x)=(\sqrt{2\pi})^{-1}\int_{-\infty}^{x}e^{-t^2/2}dt$</td></tr>
</table>

community and the machine learning community, making some fuzzy similarity measures available as potential kernels for the popular kernel-based learning methods, either on their own or in combination with existing kernels (see e.g. Maenhout et al. (2007) for an application of this type).

The second direction concerns the further exploitation of the results on the comparison of random variables. As mentioned, the approach followed here can be seen as a graded variant of the increasingly popular notion of stochastic dominance. Future research will have to clarify how these graded variants can be defuzzified in order to come up with meaningful partial orderings of random variables that are more informative than the classical notions of stochastic dominance. First results into that direction can be found in De Baets and De Meyer (2007); De Loof et al. (2006).

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