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Advances in MATHEMATICAL ECONOMICS

Volume 9

Springer

Advances in MATHEMATICAL ECONOMICS

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ISBN 4-431-34341-5 Springer-Verlag Tokyo Berlin Heidelberg New York

Printed on acid-free paper
Springer is a part of Springer Science+Business Media
springer.com

Printed in Japan

Camera-ready copy prepared from the authors' LaTeXfiles.
Printed and bound by Hirakawa Kogyosha, Japan.
SPIN: 11757900

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Adv. Math. Econ. 9, 1–23 (2006)

Advances in
MATHEMATICAL
ECONOMICS

Option on a unit-type closed-end investment fund

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Received: October 26, 2004
Revised: November 30, 2005

JEL classification: C61, G13

Mathematics Subject Classification (2000): 49L20, 60H30, 93E20

Abstract. In this paper we study options on a unit-type closed-end investment fund. These options are included among the exotic options, because the underlying asset of the options is the value process of the investment fund and therefore depends on a fund manager (= an option writer)'s action. We prove that a fair price of such option is represented as the value function of the associated stochastic exit time control problem. Using Hajek's mean comparison theorem, we find an explicit form of the fair option premium in the case of a constant volatility. We also characterize the fair option premium as a limit of a sequence of classical solutions to the associated Hamilton-Jacobi-Bellman equations with a classical Dirichlet boundary condition in the case of a diffusion market model.

Key words: capital guaranteed fund, stochastic exit time control, fair option premium, dynamic programming principle

1. Introduction

There are a considerable number of capital guaranteed funds in a financial market. One of them is a unit-type closed-end investment fund with a guarantee of refunding at least a set percentage of a principal at a redemption date. Such guarantee can be regarded as an option which is written on the investment fund and gives its holder the right to receive the difference between the set percentage of the principal and the value of the investment fund when the value of the investment fund is less than the set percentage of the principal (see Example 2.1). However the study of such options has been strangely neglected by

* I would like to express my gratitude to the anonymous referee for valuable comments and suggestions on this paper.

students in the field of the mathematical finance to the best of my knowledge. In this study, therefore, we investigate the options on the unit-type closed-end investment fund in the framework of the mathematical finance.

The feature that differentiates the options on the investment fund from the typical contingent claims is that a price fluctuation of the underlying asset (= the investment fund) depends on a trading strategy of an option writer (= a fund manager). This feature gives rise to the stochastic control problem and we will show that a fair price of the option on the investment fund is given by the value function of the associated stochastic control problem. Another option that the price fluctuation of the underlying asset depends on an action of the parties (= the option writer and/or holder) concerned about an option trading is the passport option which has been studied by Ahn *et al.* [2], Delbaen and Yor [7], Henderson [11], Henderson and Hobson [12], Hyer *et al.* [13], Nagayama [21], Shreve and Večeř [22] and the author [1]. For the case that the option holder does not have the right to redeem before maturity, we can directly apply their methods for analyzing the passport option to making an estimate of the fair price of the option on the investment fund. In particular, in §4.2 we will use the methods of Shreve and Večeř [22] to obtain an explicit form of the fair option premium in the case of a constant volatility.

We also treat the case that the option holder has the right to redeem before maturity. In this study, a criterion of an option holder's decision to redeem before maturity is set up at a purchase date of the option, and we define a redemption date as an exit time of the value process of the investment fund from some region (see (2.6)). Then we will be in need of the Dynamic Programming Principle (DPP, for short) for the stochastic exit time control problem to determine the fair option premium. In our model, although both a drift coefficient and diffusion coefficient of a controlled process depend on a control which takes value in an unbounded set, each of them is *linearly* dependent on the control. Therefore, under some conditions (see Assumption 3.1), we can adapt the arguments in §5.6 of Karatzas and Shreve [17] to our stochastic control problem in order to obtain the DPP. We will establish the DPP in §5. (We remark that we can not directly utilize the existent results for the DPP in Borkar [5] and Lions [19], because Borkar [5] was studied for only the case where the diffusion coefficient was independent of the control and it is difficult to check that the standing assumption of Lions [19, (A.2)] is satisfied with the exception of several examples.) In the dynamic programming approach to the stochastic control problem, one of most important aspects is to approximate the value function by classical solutions of the associated Hamilton-Jacobi-Bellman (HJB, for short) equations. In §4.3 we also study our control problem in this line.

This paper is organized as follows. In the next section, we specify the market model and the options on the investment fund. The main result is given in

§3. §4.2 and §4.3 present the probabilistic approach and the partial differential equational approach to the estimates of the fair option premium, respectively. The proofs of these claims are given in §5.

2. Option on investment fund

2.1 Market

Let us deal with the following model for a frictionless financial market where $N+1$ assets are traded continuously up to some fixed time horizon T. One of them is non-risky and has a price $P_0(t) = \exp\left(\int_0^t r(s)\,ds\right)$, $0 \le t \le T$. The remaining N assets are risky; their price processes $\{\widehat{P}_i(\cdot)\}_{1\le i\le N}$ are modelled by the linear stochastic differential equations

$$\frac{d\widehat{P}_i(t)}{\widehat{P}_i(t)} = \mu_i(t)\,dt + \sigma_i(t)^\top\,d\widehat{B}(t), \qquad 0 \le t \le T, \qquad \widehat{P}_i(0) = p_i > 0,$$

and their dividend rates are given by non-negative processes $\{\widehat{d}_i(\cdot)\}_{1\le i\le N}$. Here $\top$ denotes the transpose operation and $\widehat{B}(\cdot) = (\widehat{B}_1(\cdot),\ldots,\widehat{B}_N(\cdot))^\top$ is a standard N-dimensional Brownian motion on a complete probability space $(\Omega,\mathcal{F},\widehat{\mathbb{P}})$, endowed with a filtration $\mathbb{F} = \{\mathcal{F}_t\}_{0\le t\le T}$ which is the $\widehat{\mathbb{P}}$-augmentation of the filtration generated by the Brownian motion $\widehat{B}$. We assume that the risk-free rate process $r(\cdot)$, the dividend rate process $\widehat{d}(\cdot) := (\widehat{d}_1(\cdot),\ldots,\widehat{d}_N(\cdot))^\top$, the mean earning rate process $\mu(\cdot) := (\mu_1(\cdot),\ldots,\mu_N(\cdot))^\top$ and the $\mathbf{R}^N \otimes \mathbf{R}^N$-valued volatility process $\sigma(\cdot) := (\sigma_1(\cdot),\ldots,\sigma_N(\cdot))$ are $\mathbb{F}$-progressively measurable and satisfy the mild condition

$$\int_0^T \left[|r(t)| + \sum_{i=1}^N \left\{\widehat{d}_i(t) + |\mu_i(t)| + |\sigma_i(t)|^2\right\}\right] dt < \infty \qquad a.s.$$

We shall denote by $\mathcal{P}$ the set of all $\mathbf{R}^N$-valued $\mathbb{F}$-progressively measurable processes $p(\cdot)$ such that $\int_0^T |\sigma(t)p(t)|^2 dt < \infty$ *a.s.*

Furthermore we assume that the market is standard and complete.[1] Then $\sigma(t,\omega)$ is non-singular for *a.e.* $(t,\omega) \in [0,T] \times \Omega$, the process $\theta_0(\cdot) := \sigma(\cdot)^{-1}(\mu(\cdot) + \widehat{d}(\cdot) - r(\cdot)\mathbf{1}_N)$ satisfies $\int_0^T |\theta_0(s)|^2 ds < \infty$ *a.s.*, and the process

$$Z_0(t) := \exp\left[-\int_0^t \theta_0(s)^\top d\widehat{B}(s) - \frac{1}{2}\int_0^t |\theta_0(s)|^2\,ds\right], \qquad 0 \le t \le T,$$

[1] For the definition of the standard and complete market, we can refer to Chapter 1 in Karatzas and Shreve [17].

is a $(\widehat{\mathbb{P}}, \mathbb{F})$-martingale, where $\mathbf{1}_N := (1, \dots, 1)^\top \in \mathbf{R}^N$. Thanks to Girsanov's theorem,[2] therefore, the market has the risk-neutral equivalent martingale measure $\mathbb{P}$ defined by $\mathbb{P}(A) := \int_A Z_0(T,\omega)\widehat{\mathbb{P}}(d\omega)$, $A \in \mathcal{F}_T$, and $B(t) := \widehat{B}(t) + \int_0^t \theta_0(s)\, ds$, $0 \le t \le T$, is a standard N-dimensional Brownian motion on $(\Omega, \mathcal{F}_T, \mathbb{P}, \mathbb{F})$.

Let $\pi(t) = (\pi_1(t), \dots, \pi_N(t))^\top$ be the amount of money invested in corresponding assets at time t. If we denote by $\widehat{W}^{w,\pi}(t)$ the wealth of a small investor at time t, the self-financed wealth process $\widehat{W}^{w,\pi}(\cdot)$ obeys the equation

$$d\widehat{W}(t) = \sum_{i=1}^N \pi_i(t) \left\{ \widehat{d}_i(t)\, dt + \frac{d\widehat{P}_i(t)}{\widehat{P}_i(t)} \right\} + \left\{ \widehat{W}(t) - \sum_{i=1}^N \pi_i(t) \right\} \frac{dP_0(t)}{P_0(t)}, \quad (2.1)$$

$0 \le t \le T$, with an initial wealth $\widehat{W}(0) = w$. And hence, if $\pi(\cdot) \in \mathcal{P}$, the discounted self-financed wealth process $W^{w,\pi}(\cdot) := \widehat{W}^{w,\pi}(\cdot)/P_0(\cdot)$ is given by

$$W^{w,\pi}(t) = w + \int_0^t \frac{(\sigma(s)\pi(s))^\top}{P_0(s)}\, dB(s), \qquad 0 \le t \le T. \qquad (2.2)$$

Also the discounted price processes $\{P_i(\cdot) := \widehat{P}_i(\cdot)/P_0(\cdot)\}_{1 \le i \le N}$ satisfy the equations

$$\frac{dP_i(t)}{P_i(t)} = -\widehat{d}_i(t)\, dt + \sigma_i(t)^\top dB(t), \qquad 0 \le t \le T, \qquad P_i(0) = p_i > 0.$$

We set $P(\cdot) := (P_0(\cdot), P_1(\cdot), \dots, P_N(\cdot))^\top$.

2.2 Investment fund

We consider a unit-type closed-end investment fund. Let $\widehat{X}_1(t)$ be the value of the investment fund, $q(t) = (q_1(t), \dots, q_N(t))^\top$ be the proportions of $\widehat{X}_1(t)$ invested in corresponding assets, and $\delta(t)\widehat{X}_1(t)$ be the distribution of gains from investment at time t. Then, as in (2.1), the process $\widehat{X}_1(\cdot)$ obeys the equation

$$\frac{d\widehat{X}_1(t)}{\widehat{X}_1(t)} = \sum_{i=1}^N q_i(t) \left\{ \widehat{d}_i(t)\, dt + \frac{d\widehat{P}_i(t)}{\widehat{P}_i(t)} \right\} + \left\{ 1 - \sum_{i=1}^N q_i(t) - \delta(t) \right\} \frac{dP_0(t)}{P_0(t)},$$

$0 \le t \le T$, with an initial capital $\widehat{X}_1(0) = x > 0$. Hence if we denote by $\mathcal{D}$ the class of all $\mathbb{F}$-progressively measurable processes $\delta(\cdot)$ such that $0 \le \delta(\cdot) < 1$

[2] See Theorem 3.5.1 of Karatzas and Shreve [16].

a.e., and if $u(\cdot) = (\delta(\cdot), q(\cdot)) \in \mathcal{D} \times \mathcal{P}$, then the discounted value process $X_1^{x,u}(\cdot) := \widehat{X}_1^{x,u}(\cdot)/P_0(\cdot)$ of the investment fund satisfies the equation

$$\frac{dX_1(t)}{X_1(t)} = -\delta(t)\,dt + (\sigma(t)q(t))^\top dB(t), \quad 0 \le t \le T, \quad X_1(0) = x, \quad (2.3)$$

and $X_1^{x,u}(\cdot) > 0$ *a.e.* We assume that the fund manager can choose the strategy $u(\cdot) = (\delta(\cdot), q(\cdot))$ from among some class $\mathcal{U} \subset \mathcal{D} \times \mathcal{P}$.

2.3 Option

Let us consider the following options. A financial institution is planning to sell the investment fund $\mathcal{U}$ with an *option* whose holder has the right to receive the payment

$$P_0(\tau_0) f(\tau_0, X_0^u(\tau_0), X_1^{x,u}(\tau_0), P(\tau_0)) \qquad \text{at time } \tau_0 \in [0, T], \qquad (2.4)$$

where f is a nonnegative continuous function on $[0,T] \times [0,\infty) \times (0,\infty)^{N+2}$ with the properties $f(t,\cdot,x_1,p)$ (resp. $f(t,x_0,\cdot,p)$) is non-increasing for each $(t,x_1,p) \in [0,T]\times(0,\infty)^{N+2}$ (resp. $(t,x_0,p) \in [0,T]\times[0,\infty)\times(0,\infty)^{N+1}$),

$$X_0^u(t) := \int_0^t \delta(s) X_1^{x,u}(s) l_0(P_0(s))\, ds,$$

l_0 is a nonnegative continuous function on $(0,\infty)$, a call date $\tau_0 = \tau_0^{x,p,u}$ before maturity T is

$$\tau_0 := \inf\{t \ge 0 : (X_0^u(t), X_1^{x,u}(t), P(t)) \notin O\} \wedge T,$$

and O is an open subset of $\mathbf{R}^{N+3}$.

Example 2.1 (Capital Guaranteed Funds). The discounted payoff function corresponding to the capital guaranteed funds is represented as

$$\begin{aligned} & f(t, X_0^u(t), X_1^{x_1,u}(t), P(t)) \\ &= \frac{1}{l_1(P_0(t))}\left(ax_1 - \{X_0^u(t) + l_1(P_0(t)) X_1^{x_1,u}(t)\}\right)^+ \end{aligned} \qquad (2.5)$$

for some constant $a > 0$ and a positive function $l_1 \in C(0,\infty)$. In §4 we will study for the following cases:

(A) In the case $l_0(\cdot) \equiv 0$ and $l_1(\cdot) \equiv 1$ (resp. $l_1(p_0) = p_0$), the investment fund with the option (2.5) guarantees to refund at least $100a\%$ of the *real* (resp. *nominal*) value of the principal x_1, since

$$\max\{ax_1, X_1^{x_1,u}(t)\} = X_1^{x_1,u}(t) + (ax_1 - X_1^{x_1,u}(t))^+.$$

(B) In the case $l_0(\cdot) = l_1(\cdot) \equiv 1$ (resp. $l_0(p_0) = l_1(p_0) = p_0$), the investment fund with the option (2.5) guarantees that the sum of the redemption value and the total distribution of gains from investment is not less than ax_1 in *real* (resp. *nominal*) terms.

In relation to the examples above, we are interested in the following call date:

$$\tau_0 := \inf\left\{t \geq 0 : Y^u(t) := X_0^u(t) + l_1(P_0(t))X_1^{x_1,u}(t) \leq bx_1\right\} \wedge T \quad (2.6)$$

for $0 \leq b < a \wedge 1$. In §4 we will estimate the fair option premium for such cases by making use of Hajek's mean comparison theorem and the standard theory of the partial differential equations (PDEs, for short).

3. Fair price

Our purpose is to estimate the fair price of the option (2.4). The objective of the option writer is to find a pair of strategies (π, u) such that

$$W^{w,\pi}(\tau_0) \geq f\big(\tau_0, X_0^u(\tau_0), X_1^{x,u}(\tau_0), P(\tau_0)\big) \qquad a.s., \qquad (3.1)$$

where w is the amount that the writer receives from the holder at time $t = 0$. The inequality (3.1) says that the writer's wealth starting with the initial wealth w will have grown by the call date τ_0 enough to cover the payment $P_0(\tau_0)f\big(\tau_0, X_0^u(\tau_0), X_1^{x,u}(\tau_0), P(\tau_0)\big)$ which he has to provide the holder at τ_0. We remark that the aim of replicating the option does not conflict with an obligation to seek to increase the value of investment fund because $f(t, \cdot, x_1, p)$ and $f(t, x_0, \cdot, p)$ are non-increasing. Therefore the *upper hedging price*, which is the least initial amount w that enables the writer to achieve (3.1), is denoted by

$$h_{up}(x,p) = \inf\{w \in H_{up}(x,p)\}$$

for each $(x,p) \in (0,\infty)^{N+2}$, where

$$H_{up}(x,p) = \left\{ w \geq 0 \;\middle|\; \begin{array}{l} \exists(\pi, u) \in \mathcal{P} \times \mathcal{U} \quad s.t. \\ \text{(i) } \exists\beta \in \mathbf{R} \;\; s.t. \;\; \mathbb{P}\{W^{0,\pi}(t) \geq \beta, \; t \in [0,T]\} = 1. \\ \text{(ii) (3.1) is achieved.} \end{array} \right\}$$

On the other hands, the option holder desires to find a portfolio strategy π^u, which is chosen according to a strategy u making up the investment fund, such that

$$W^{-w,\pi^u}(\tau_0) + f\big(\tau_0, X_0^u(\tau_0), X_1^{x,u}(\tau_0), P(\tau_0)\big) \geq 0 \qquad a.s., \qquad (3.2)$$

where $-w$ is the debt that he incurred at time $t = 0$ by purchasing the option. The inequality (3.2) means that if he adopts the strategy π^u then the payment

$P_0(\tau_0)f\big(\tau_0, X_0^u(\tau_0), X_1^{x,u}(\tau_0), P(\tau_0)\big)$ at the call date τ_0 make it possible for him to cover the debt $-w$. Therefore the *lower hedging price*, which is the largest amount w that enables the holder to achieve (3.2), is denoted by

$$h_{low}(x,p) = \sup\{w \in H_{low}(x,p)\}$$

for each $(x,p) \in (0,\infty)^{N+2}$, where

$$H_{low}(x,p) = \left\{ w \geq 0 \;\middle|\; \begin{array}{l} \forall u \in \mathcal{U}, \quad \exists \pi^u \in \mathcal{P} \quad s.t. \\ \text{(i) } W^{0,\pi^u}(\cdot) \text{ is a } (\mathbb{P},\mathbb{F})\text{-supermartingale.} \\ \text{(ii) } (3.2) \text{ is achieved.} \end{array} \right\}.$$

To ensure $h_{up} = h_{low}$, we are now in a position to make some conditions.

Assumption 3.1.

(i) *For each $u \in \mathcal{U}$, there exists $m > 1$ such that*

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} f(t, X_0^u(t), X_1^{x,u}(t), P(t))^m \right] < \infty.$$

(ii) *There exists a sequence $\{f_n\}_{n=1}^{\infty}$ of $C^{1,1,2}([0,T] \times [0,\infty) \times (0,\infty)^{N+2})$ such that*

$$\lim_{n \to \infty} \|f - f_n\|_\infty = 0 \qquad \textit{and} \qquad \mathbb{E}\left[\int_0^T |G^u f_n(t)|\, dt \right] < \infty$$

for each $u \in \mathcal{U}$ and $n \geq 1$, where $\int_0^\bullet G^u f_n(t)\, dt$ is the finite variation process in the canonical decomposition of the semimartingale $f_n(\cdot, X_0^u(\cdot), X_1^{x,u}(\cdot), P(\cdot))$.

Then we have the main result:

Theorem 3.2. *Under Assumption 3.1, $h_{up}(x,p) = h_{low}(x,p) = V(x,p)$ for each $(x,p) \in (0,\infty)^{N+2}$, namely, the fair price of the option* (2.4) *is given by $V(x,p)$, where*

$$V(x,p) = \inf_{u \in \mathcal{U}} \mathbb{E}\big[f\big(\tau_0, X_0^u(\tau_0), X_1^{x,u}(\tau_0), P^p(\tau_0)\big)\big]. \tag{3.3}$$

4. Estimates for Example 2.1

In this section, we confine our attention to the problem of estimating the fair price of the option (2.5) with the call date (2.6). In order to begin addressing this problem, we make sure that the process $Y^{x_1,u}(\cdot) = X_0^u(\cdot) + l_1(P_0(\cdot))X_1^{x_1,u}(\cdot)$ obeys the equations:

In the case (A): $\dfrac{dY(t)}{Y(t)} = \big(-\delta(t) + l_1'(P_0(t))r(t)\big)\,dt + \big(\sigma(t)q(t)\big)^{\top} dB(t),$
$$\tag{4.1}$$
In the case (B): $\dfrac{dY(t)}{l_1(P_0(t))X_1(t)} = l_1'(P_0(t))r(t)\,dt + \big(\sigma(t)q(t)\big)^{\top} dB(t),$

respectively.

4.1 Zero-premium option

To start with, we consider for the case $u_0 \equiv 0 \in \mathcal{U}$. We first see that

$$V(x_1, p) = x_1(a-1)^+, \qquad (x_1, p) \in (0,\infty)^{N+2}$$

for the case $l_1(\cdot) \equiv 1$. Indeed, since $Y^{x_1,u}(\cdot)$ is a $(\mathbb{P},\mathbb{F})$-supermartingale, Jensen's inequality gives

$$\mathbb{E}\Big[\big(ax_1 - Y^{x_1,u}(\tau_0^u)\big)^+\Big] \geq x_1(a-1)^+ = \mathbb{E}\Big[\big(ax_1 - Y^{x_1,u_0}(\tau_0^{u_0})\big)^+\Big]$$

for all $u \in \mathcal{U}$.

Further, we suppose that $T^{-1}\log a \leq r(t)$ *a.s.* for every $t \in [0,T]$. Then, for the case $l_1(p_0) = p_0$, we obtain $V(x_1,p) = 0$ because $Y^{x_1,u_0}(T) = x_1P_0(T) \geq a\,x_1$.

Consequently, we have the fact: *If* $0 \in \mathcal{U}, 0 < a \leq 1$ *and the risk-free rate* $r(\cdot)$ *is non-negative, then the fair price of the option* (2.5) *with the call date* (2.6) *is equal to zero.*

Remark 4.1. The condition $0 \in \mathcal{U}$ enables the fund manager to invest all the funds in risk-less asset. And then he may invest all the funds in risk-less asset to avoid the advanced redemption, even if he knows that it may be against the beneficiary's own interests. For reasons mentioned above, it seems reasonable to assume that $0 \notin \mathcal{U}$. In §4.3, therefore, we will develop the arguments under the condition $0 \notin \mathcal{U}$.

4.2 Constant volatility

Next we consider for the case $l_0(\cdot) \equiv 0$ and $l_1(\cdot) \equiv 1$, i.e.,

$$f\big(t, X_0^u(t), X_1^{x_1,u}(t), P(t)\big) = \big(ax_1 - X_1^{x_1,u}(t)\big)^+,$$
$$\tau_0 = \inf\big\{t \geq 0 : X_1^{x_1,u}(t) \leq bx_1\big\} \wedge T$$

in the market with a constant volatility matrix $\sigma(\cdot) \equiv \sigma \in \mathbf{R}^N \otimes \mathbf{R}^N$. We also assume that